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Observables as Twist Anomaly in Vacuum String Field Theory

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Abstract

We reveal a novel mathematical structure in physical observables, the mass of tachyon fluctuation mode and the energy density, associated with a classical solution of vacuum string field theory constructed previously [hep-th/0108150]. We find that they are expressed in terms of quantities which apparently vanish identically due to twist even-odd degeneracy of eigenvalues of a Neumann coefficient matrix defining the three-string interactions. However, they can give non-vanishing values because of the breakdown of the degeneracy at the edge of the eigenvalue distribution. We also present a general prescription of correctly simplifying the expressions of these observables. Numerical calculation of the energy density following our prescription indicates that the present classical solution represents the configuration of two D25-branes.

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1 Introduction

Vacuum string field theory (VSFT) [1, 2, 3, 4] has been proposed as a string field theory expanded around the tachyon vacuum [5, 6, 7]. The action of VSFT is simply given by that of ordinary cubic string field theory (CSFT) with its BRST operator replaced by $\mathcal{Q}$ linear in the ghost coordinate:

$$\mathcal{Q} = c_0 + \sum_{n\geq 1} f_n \left(c_n + (-1)^n c_n^\dagger \right) . \tag{1.1}$$

Since the cohomology of $\mathcal{Q}$ is trivial, VSFT expanded around the trivial configuration $\Psi = 0$ contains no physical open string excitations at all. Therefore, VSFT around $\Psi = 0$ is believed to describe pure closed string theory though no direct proof for this expectation has been given yet.

Another problem concerning VSFT is to show that it has classical solutions representing Dp-branes, in particular, D25-bane. The energy density of these solutions relative to the trivial one must be equal to the Dp-brane tension. The perturbation expansion around the D25-brane solution must reproduce the ordinary bosonic open string theory.

The fact that the BRST operator $\mathcal{Q}_{\mathrm{B}}$ of VSFT consists purely of the ghost coordinate makes it easier to solve its classical equation of motion. First, we can consider solutions factorized into the matter part and the ghost one. Second, for this type solution, the equation of motion for the matter part implies that it is a kind of projection operator [8, 9, 10, 11, 12, 13]. Using these facts, the matter part of the solutions have been obtained and the ratio of the energy densities of two Dp-brane solutions have been found to reproduce the expected tension ratio [1, 2]. In these analyses they assumed that the ghost part is common among Dp-brane solutions for all p, and hence it was unnecessary to know the explicit form of the ghost part.

However, for studying whether the energy density of Dp-brane solution itself, instead of the ratio, is equal to the correct one, and whether the perturbation expansion around the solutions reproduce the known open string theories, we have to obtain the solutions including their ghost parts. In [14], they constructed a translationally and Lorentz invariant classical solution of VSFT including the ghost part, and analyzed the fluctuation spectrum and the energy density of the solution. The mass of the tachyon fluctuation mode and the ratio of the energy density to the D25-brane tension are given in closed forms using the Neumann coefficients defining the three-string interactions. They calculated these two quantities numerically using the level truncation. Though the tachyon mass was correctly reproduced, the calculation of the energy density did not give the expected value of the D25-brane tension.

In this paper, we shall unmask beautiful mathematical structures of physical observables such as the tachyon mass and the ratio of the energy density to the D25-brane tension obtained in [14]. In [14], they gave the tachyon mass squared and the ratio in the form $-\ln 2/G$ and $\pi^2/(3\ln^3 2)\exp(-6H)$, respectively, using G and H which are expressed in closed forms using Neumann coefficients. We find that both G and H are quantities which vanish identically if we use the known identities among the Neumann coefficients. On the other hand, numerical calculation of these quantities gave non-vanishing results. We identify the origin of this paradox. We argue that both G and H are quantities similar to the chiral index of fermions [15] or the Witten index in supersymmetric theories [16]. They almost vanish because of the degeneracy of eigenvalues of a Neumann coefficient matrix due to world-sheet twist transformation. Non-vanishing values of G and H come from the breakdown of degeneracy at the edge of the eigenvalue distribution. Therefore we call this phenomenon twist anomaly.

As we mentioned above, it is dangerous to naively use the identities among the Neumann coefficients to simplify the expressions of G and H. We also present a general prescription for allowed deformation of these quantities by taking into account the singularities at the edge of the eigenvalue distribution. By respecting the above lessons, we reexamine the energy density of the classical solution. Our numerical calculation indicates that the classical solution of [14] represents two D25-branes if there are no other subtle points.

The organization of the rest of this paper is as follows. In sec. 2, after presenting the elements of the VSFT action, in particular, the identities among the Neumann coefficients, we summarize the classical solution presented in [14]. In sec. 3, we point out that the quantity G giving the tachyon mass vanishes if we naively use the identities, and then resolve the paradox. We also present a general prescription of allowed deformations for G and H. In sec. 4, we reexamine the energy density of the solution. In the final section, we summarize the paper and discuss future problems.

2 VSFT and its classical solution

In this section we shall first summarize basic elements of VSFT, in particular, the Neumann coefficient algebra, and review its translationally invariant classical solution given in [14].

2.1 VSFT action

The action of VSFT is given by [1, 2, 4]

$$\mathcal{S}[\Psi] = -K\biggl(\frac{1}{2}\Psi\cdot\mathcal{Q}\Psi + \frac{1}{3}\Psi\cdot(\Psi*\Psi)\biggr), \tag{2.1}$$

where the front factor K is a constant. The BRST operator $\mathcal{Q}$ around the tachyon vacuum is given by a purely ghost form (1.1), and satisfies the nilpotency and the Leibniz rule on the $*$-product. The three-string vertex defining the $*$-product is the same as in the ordinary CSFT and is given in the momentum representation for the center-of-mass x^μ as [17, 18, 19, 20, 21, 22]

$$\begin{aligned} |V\rangle_{123} = \exp\left(-\frac{1}{2}\boldsymbol{A}^\dagger C\mathcal{M}_3\boldsymbol{A}^\dagger - \boldsymbol{A}^\dagger\boldsymbol{V} - \frac{1}{2}V_{00}(A_0)^2 + (\text{ghost part})\right)|0\rangle_{123} \\ \times(2\pi)^{26}\delta^{26}\left(p_1+p_2+p_3\right), \end{aligned} \tag{2.2}$$

with various quantities defined by

$$\begin{gathered} \boldsymbol{A} = \begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \\ a_n^{(3)} \end{pmatrix}, \quad A_0 = \begin{pmatrix} a_0^{(1)} \\ a_0^{(2)} \\ a_0^{(3)} \end{pmatrix}, \quad \mathcal{M}_3 = \begin{pmatrix} M_0 & M_+ & M_- \\ M_- & M_0 & M_+ \\ M_+ & M_- & M_0 \end{pmatrix}, \\ \boldsymbol{V} = \begin{pmatrix} \boldsymbol{v}_0 & \boldsymbol{v}_+ & \boldsymbol{v}_- \\ \boldsymbol{v}_- & \boldsymbol{v}_0 & \boldsymbol{v}_+ \\ \boldsymbol{v}_+ & \boldsymbol{v}_- & \boldsymbol{v}_0 \end{pmatrix} A_0, \quad V_{00} = \frac{1}{2}\ln\left(\frac{3^3}{2^4}\right). \end{gathered} \tag{2.3}$$

The boldface letters, $\boldsymbol{A}$ and $\boldsymbol{V}$, are the vectors in the level number space. The matter oscillator $a_n^{(r)\mu}$ $(n \geq 1)$ satisfies the commutation relation,

$$[a_n^{(r)\mu}, a_m^{(s)\nu\dagger}] = \eta^{\mu\nu}\delta_{nm}\delta^{rs}, \tag{2.4}$$

and $a_0^{(r)}$ is related to the center-of-mass momentum of the string r, $p_r = -i\partial/\partial x_r$, by $a_0^{(r)} = \sqrt{2}\,p_r$ (we are adopting the convention of $\alpha' = 1$). The real and symmetric matrices $(M_0)_{nm}$ and $(M_\pm)_{nm}$ and the real vectors $(\boldsymbol{v}_0)_n$ and $(\boldsymbol{v}_\pm)_n$ in the level number space are essentially the Neumann coefficient matrices (see [14] for their relation to the conventional Neumann coefficients). Finally, C is the twist matrix defined by

$$C_{nm} = (-1)^n\delta_{nm}, \qquad (n, m \geq 1). \tag{2.5}$$

It should be noted that the inner products in the exponent of (2.2) are those in both the infinite dimensional level number space and the three dimensional space of the strings $r = 1, 2, 3$ (we have omitted the transpose symbol for the vectors multiplying from the left).

In the rest of this subsection, we shall summarize the algebras of the Neumann coefficients M_α and $\boldsymbol{v}_\alpha$ ($\alpha = 0, \pm$). First, the twist transformation property of the vertex,

$$\Omega_1\Omega_2\Omega_3|V\rangle_{123} = |V\rangle_{321}, \tag{2.6}$$

is translated to the following for the Neumann coefficients:

$$CM_0C = M_0, \quad CM_\pm C = M_\mp, \quad C\boldsymbol{v}_0 = \boldsymbol{v}_0, \quad C\boldsymbol{v}_\pm = \boldsymbol{v}_\mp. \tag{2.7}$$

Here, Ω_r is the twist operator on the Fock space of the string r:

$$\Omega\, a_n\Omega^{-1} = C_{nm}a_m, \qquad \Omega|0\rangle = |0\rangle. \tag{2.8}$$

Next, they enjoy the following linear relations:

$$M_0 + M_+ + M_- = 1, \quad \boldsymbol{v}_0 + \boldsymbol{v}_+ + \boldsymbol{v}_- = 0. \tag{2.9}$$

Therefore, let us take (M_0, M_1) and $(\boldsymbol{v}_0, \boldsymbol{v}_1)$ with M_1 and $\boldsymbol{v}_1$ defined by

$$M_1 = M_+ - M_-, \qquad \boldsymbol{v}_1 = \boldsymbol{v}_+ - \boldsymbol{v}_-, \tag{2.10}$$

as independent quantities. Note that M_0 and $\boldsymbol{v}_0$ are twist-even, while M_1 and $\boldsymbol{v}_1$ are twist-odd:

$$CM_1C = -M_1, \quad C\boldsymbol{v}_1 = -\boldsymbol{v}_1. \tag{2.11}$$

Then, M_0, M_1, $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ are known to satisfy the following non-linear identities [20, 21, 23]:

$$[M_0, M_1] = 0, \tag{2.12}$$
$$M_1^2 = (1 - M_0)(1 + 3M_0), \tag{2.13}$$
$$3(1 - M_0)\boldsymbol{v}_0 + M_1\boldsymbol{v}_1 = 0, \tag{2.14}$$
$$3M_1\boldsymbol{v}_0 + (1 + 3M_0)\boldsymbol{v}_1 = 0, \tag{2.15}$$
$$\frac{9}{4}\boldsymbol{v}_0^2 + \frac{3}{4}\boldsymbol{v}_1^2 = 2\,V_{00}. \tag{2.16}$$

2.2 Classical solution of VSFT

Now let us proceed to reviewing the construction of a translationally and Lorentz invariant classical solution $\Psi_{\rm c}$ to the equation of motion of the VSFT action (2.1) [24, 2, 14]:

$$\mathcal{Q}\Psi_{\rm c} + \Psi_{\rm c} * \Psi_{\rm c} = 0. \tag{2.17}$$

The solution is expected to represent a space-time filling D25-brane.

We adopt the Siegel gauge for $\Psi_{\rm c}$, $|\Psi_{\rm c}\rangle = b_0|\phi_{\rm c}\rangle$, and assume the following form for $|\phi_{\rm c}\rangle$:

$$|\phi_{\rm c}\rangle = \mathcal{N}_{\rm c} \exp\biggl(-\frac{1}{2}\sum_{n,m\geq 1} a_n^\dagger (CT)_{nm} a_m^\dagger + \sum_{n,m\geq 1} c_n^\dagger (C\widetilde{T})_{nm} b_m^\dagger\biggr)|0\rangle, \tag{2.18}$$

where T_{nm} and $\widetilde{T}_{nm}$ are unknown real matrices and $\mathcal{N}_{\rm c}$ is the normalization factor. We assume further that the state $|\phi_{\rm c}\rangle$ is twist invariant, $\Omega|\phi_{\rm c}\rangle = |\phi_{\rm c}\rangle$, and hence T_{nm} and $\widetilde{T}_{nm}$ satisfy the matrix equations

$$CTC = T, \qquad C\widetilde{T}C = \widetilde{T}. \tag{2.19}$$

Then, $\Psi_{\rm c}$ solves the equation of motion (2.17) provided the following two conditions are satisfied:

- T and $\widetilde{T}$ satisfy

  $$T = M_0 + (M_+, M_-)(1 - T\mathcal{M})^{-1} T \begin{pmatrix} M_+ \\ M_- \end{pmatrix}, \tag{2.20}$$

  with

  $$\mathcal{M} = \begin{pmatrix} M_0 & M_+ \\ M_- & M_0 \end{pmatrix}, \tag{2.21}$$

  and the same one with all the matrices replaced by the tilded ones for the ghost oscillators, respectively. The matrix T on the RHS of (2.20) should read $\mathrm{diag}(T, T)$.

- The normalization factor $\mathcal{N}_{\rm c}$ is given by

  $$\mathcal{N}_{\rm c} = -\left[\det(1 - T\mathcal{M})\right]^{13}\left[\det(1 - \widetilde{T}\widetilde{\mathcal{M}})\right]^{-1}. \tag{2.22}$$

The arbitrary coefficient f_n in the BRST operator $\mathcal{Q}$ (1.1) is not a quantity which is given apriori, but rather it is uniquely fixed by the requirement that the there exist a Siegel gauge solution assumed above. See [14] for details.*

Eq. (2.20) for T has been solved in [24, 2], and we shall summarize the points in obtaining the solution. Let us assume that T commutes with the matrices M_α:

$$[T, M_\alpha] = 0, \qquad (\alpha = 0, \pm). \tag{2.23}$$

*A concise expression of the coefficient f_n is given in [23].

Using the formulas (2.12) and (2.13) for M_α and, in particular,

$$(1-T\mathcal{M})^{-1} = \left(1-2M_0T+M_0T^2\right)^{-1}\begin{pmatrix}1-TM_0 & TM_+\\ TM_- & 1-TM_0\end{pmatrix}, \tag{2.24}$$

eq. (2.20) is reduced to

$$(T-1)\Big(M_0T^2-(1+M_0)T+M_0\Big)=0. \tag{2.25}$$

We do not adopt the solution $T=1$ which corresponds to the identity state, and take a solution to

$$M_0T^2-(1+M_0)T+M_0=0. \tag{2.26}$$

As a solution to (2.26) we take

$$T=\frac{1}{2M_0}\left(1+M_0-\sqrt{(1-M_0)(1+3M_0)}\right). \tag{2.27}$$

The matrix square root in (2.27) is defined as the positive branch of the square root of the eigenvalues of the symmetric matrix $(1-M_0)(1+3M_0)$.[†] It has been claimed by numerical comparison that the matter part of the solution given by the present T is equal to the sliver state constructed by CFT arguments [25, 2].

3 Tachyon mass as twist anomaly

In this section, we shall reexamine the mass of the tachyon fluctuation mode around Ψ_c obtained in [14]. We shall find that it has an interesting interpretation as a kind of anomaly. For this purpose, we shall first summarize the construction of the tachyon wave function.

3.1 Tachyon fluctuation mode

In [14] fluctuation spectrum around the classical solution Ψ_c was also studied. Expanding the original string field Ψ in VSFT as

$$\Psi=\Psi_\mathrm{c}+\Phi, \tag{3.1}$$

with Φ being the fluctuation, the VSFT action (2.1) is expressed as

$$\mathcal{S}[\Psi]=\mathcal{S}[\Psi_\mathrm{c}]-K\left(\frac{1}{2}\Phi\cdot\mathcal{Q}_\mathrm{B}\Phi+\frac{1}{3}\Phi\cdot(\Phi*\Phi)\right), \tag{3.2}$$

[†]See sec. 3 for the eigenvalue distribution of M_0.

where $\mathcal{Q}_{\mathrm{B}}$ is defined by

$$\mathcal{Q}_{\mathrm{B}}\Phi = \mathcal{Q}\,\Phi + \Psi_{\mathrm{c}} * \Phi + \Phi * \Psi_{\mathrm{c}}. \tag{3.3}$$

The new BRST operator $\mathcal{Q}_{\mathrm{B}}$ also satisfies the nilpotency and the Leibniz rule on the $*$-product.

We shall recapitulate the construction of tachyon wave function Φ_{t} given in [14]. It is a scalar solution to

$$\mathcal{Q}_{\mathrm{B}}\Phi_{\mathrm{t}} = 0, \tag{3.4}$$

and carries center-of-mass momentum $p^2 = 1$. We take again the Siegel gauge for Φ_{t}, $|\Phi_{\mathrm{t}}\rangle = b_0|\phi_{\mathrm{t}}\rangle$, and assume the following form for $|\phi_{\mathrm{t}}\rangle$:

$$|\phi_{\mathrm{t}}\rangle = \frac{\mathcal{N}_{\mathrm{t}}}{\mathcal{N}_{\mathrm{c}}}\exp\Bigg(-\sum_{n\geq 1} t_n a_n^\dagger a_0\Bigg)|\phi_{\mathrm{c}}\rangle. \tag{3.5}$$

Though not written explicitly, $|\phi_{\mathrm{t}}\rangle$ carries non-vanishing momentum in contrast with $|\phi_{\mathrm{c}}\rangle$ which is translationally invariant. Since $|\phi_{\mathrm{t}}\rangle$ is twist invariant, the vector $\boldsymbol{t}$ satisfies

$$C\boldsymbol{t} = \boldsymbol{t}. \tag{3.6}$$

The normalization factor $\mathcal{N}_{\mathrm{t}}$ for $|\phi_{\mathrm{t}}\rangle$ will be fixed later. Then, the wave equation (3.4) holds for the present Φ_{t} if the vector $\boldsymbol{t}$ satisfies

$$\boldsymbol{t} = \boldsymbol{v}_0 - \boldsymbol{v}_+ + (M_+, M_-)(1 - T\mathcal{M})^{-1} T \begin{pmatrix} \boldsymbol{v}_+ - \boldsymbol{v}_- \\ \boldsymbol{v}_- - \boldsymbol{v}_0 \end{pmatrix} + (M_+, M_-)(1 - T\mathcal{M})^{-1} \begin{pmatrix} 0 \\ \boldsymbol{t} \end{pmatrix}, \tag{3.7}$$

and the center-of-mass momentum p_μ is subject to $p^2 = -m_{\mathrm{t}}^2$ with the tachyon mass m_{t} given by

$$-m_{\mathrm{t}}^2 = \frac{\ln 2}{G}, \tag{3.8}$$

in terms of G defined by

$$\begin{aligned} G = 2V_{00} &+ (\boldsymbol{v}_- - \boldsymbol{v}_+, \boldsymbol{v}_+ - \boldsymbol{v}_0)(1 - T\mathcal{M})^{-1} T \begin{pmatrix} \boldsymbol{v}_+ - \boldsymbol{v}_- \\ \boldsymbol{v}_- - \boldsymbol{v}_0 \end{pmatrix} \\ &+ 2(\boldsymbol{v}_- - \boldsymbol{v}_+, \boldsymbol{v}_+ - \boldsymbol{v}_0)(1 - T\mathcal{M})^{-1} \begin{pmatrix} 0 \\ \boldsymbol{t} \end{pmatrix} + (0, \boldsymbol{t})\mathcal{M}(1 - T\mathcal{M})^{-1} \begin{pmatrix} 0 \\ \boldsymbol{t} \end{pmatrix}. \end{aligned} \tag{3.9}$$

We have to solve (3.7) for the vector $\boldsymbol{t}$ to check whether m_{t}^2 given by (3.8) really reproduces the correct value of the tachyon mass, $-m_{\mathrm{t}}^2 = 1$. First, eq. (3.7) was solved to give

$$\boldsymbol{t} = 3(1 + T)(1 + 3M_0)^{-1}\boldsymbol{v}_0. \tag{3.10}$$

Putting this solution into (3.9) the following expression for G was obtained:

$$G = \frac{9}{4}\boldsymbol{v}_0 \frac{1}{1+3M_0}\widehat{G}\boldsymbol{v}_0 - \frac{1}{4}\boldsymbol{v}_1 \frac{1}{1-M_0}\widehat{G}\boldsymbol{v}_1, \tag{3.11}$$

with

$$\widehat{G} = \frac{-(1+3M_0)^2(1-2M_0) + (1-3M_0)\sqrt{(1-M_0)(1+3M_0)}}{2M_0(1+3M_0)}. \tag{3.12}$$

The square root in (3.12) came from that in T (2.27) and hence the same branch should be taken.

3.2 Reexamination of the tachyon mass

In [14] they used the level truncation to evaluate the quantity G numerically and found that it reproduces to high precision the expected value $G = \ln 2$. However, by using the formulas (2.12), (2.13) and (2.14) in the expression (3.11), we can show that G vanishes identically! In fact, plugging

$$\boldsymbol{v}_0 = -\frac{1}{3(1-M_0)}M_1\boldsymbol{v}_1, \tag{3.13}$$

obtained from (2.14) into the first term on the RHS of (3.11) and using the commutativity (2.12) and the formula (2.13), we find that the two terms on the RHS cancel each other. Note that this cancellation occurs for any $\widehat{G}$ commutative with M_0 and M_1. Numerical analysis of G gives a finite and non-vanishing result, while we can analytically show that the same quantity vanishes identically. Where does this paradox come from?

As a preparation for understanding the origin of the paradox, we shall mention the eigenvalue distribution of M_0. We have evaluated the eigenvalues of M_0 numerically by the level truncation, namely, by cutting off the size of the infinite dimensional matrix to a finite $L \times L$ one. The result is as follows. First, all the eigenvalues of M_0 are negative and most of them are close to zero. Next, table 1 shows the three smallest eigenvalues for various values of the cutoff L. The smallest eigenvalue λ_1 at $L = \infty$ is the result of extrapolation by a fitting function of the form $\sum_{k=0}^{10} c_k (\ln L)^{-k}$. From this analysis and taking the arguments below into account in advance, it is very likely that the smallest eigenvalue of M_0 converges to $-1/3$ in the limit $L \to \infty$. The eigenvalue distribution of M_0 would be continuous in the range $(-1/3, 0)$.

Let us return to the paradox. To identify the origin of the paradox and interpret it as twist anomaly, let us carry out the following formal argument by assuming that the eigenvalue

L	λ_1	λ_2	λ_3
50	-0.28120	-0.17637	-0.08862
100	-0.28963	-0.19607	-0.11006
150	-0.29367	-0.20618	-0.12190
200	-0.29621	-0.21277	-0.12994
250	-0.29802	-0.21758	-0.13598
300	-0.29940	-0.22133	-0.14077
400	-0.30143	-0.22691	-0.14808
500	-0.30288	-0.23098	-0.15354
600	-0.30399	-0.23415	-0.15787
800	-0.30563	-0.23888	-0.16446
1000	-0.30681	-0.24234	-0.16937
∞	-0.33342		

Table 1: The three smallest eigenvalues of M_0 for various values of the cutoff L.

distribution of M_0 is in the range $(-1/3, 0)$ and for simplicity that it is discrete. Then, note first that the eigenvalues λ of M_0 are two-fold degenerate except at $\lambda = -1/3$. In fact, as seen by using the formulas (2.12) and (2.13), the twist-even and odd eigenvectors $\boldsymbol{u}_\lambda^{(\pm)}$ of M_0 corresponding to the eigenvalue λ and satisfying

$$M_0 \boldsymbol{u}_\lambda^{(\pm)} = \lambda \boldsymbol{u}_\lambda^{(\pm)}, \quad C\boldsymbol{u}_\lambda^{(\pm)} = \pm \boldsymbol{u}_\lambda^{(\pm)}, \quad \boldsymbol{u}_\lambda^{(s)} \cdot \boldsymbol{u}_{\lambda'}^{(s')} = \delta_{s,s'}\delta_{\lambda,\lambda'}, \tag{3.14}$$

are related by

$$\boldsymbol{u}_\lambda^{(\pm)} = \frac{1}{\sqrt{(1-\lambda)(1+3\lambda)}} M_1 \boldsymbol{u}_\lambda^{(\mp)}. \tag{3.15}$$

However, for $\lambda = -1/3$ degeneracy does not occur in general since we have $M_1 \boldsymbol{u}_{\lambda=-1/3}^{(\pm)} = 0$ due to (2.13). Because the eigenvector corresponding to the lowest eigenvalue λ_1 in table 1 is twist-odd,[‡] we assume that the eigenvector corresponding to $\lambda = -1/3$ is twist-odd and there is no corresponding twist-even eigenvector. Then, we expand $\boldsymbol{v}_1$ and $\boldsymbol{v}_0$ in terms of $\{\boldsymbol{u}_\lambda^{(-)}\}$ and $\{\boldsymbol{u}_\lambda^{(+)}\}$, respectively:

$$\boldsymbol{v}_1 = \sum_\lambda A_\lambda \boldsymbol{u}_\lambda^{(-)}, \quad \boldsymbol{v}_0 = -\frac{1}{3} \sum_{\lambda \neq -1/3} \sqrt{\frac{1+3\lambda}{1-\lambda}} A_\lambda \boldsymbol{u}_\lambda^{(+)}, \tag{3.16}$$

where the coefficients in $\boldsymbol{v}_0$ has been determined by using (2.14). Plugging these expansions

[‡]Numerical evaluation of the eigenvectors of M_0 shows that they are alternatively twist-even and odd as the eigenvalue increases.

into G (3.11), we obtain

$$G = \left(\sum_{\lambda \neq -1/3} - \sum_{\lambda} \right) \frac{9}{4} \frac{\widehat{G}(\lambda)}{1 + 3\,\lambda} A_\lambda^2, \tag{3.17}$$

where $\widehat{G}(\lambda)$ is given by (3.12) with the matrix M_0 replaced by its eigenvalue λ.

Eq. (3.17) implies that G is a quantity similar to the chiral index of fermions [15] or the Witten index in supersymmetric theories [16]. It almost vanishes due to cancellation between twist-odd and even contributions. However, owing to the mismatch at $\lambda = -1/3$, G can be non-vanishing. Therefore, we call such phenomenon twist anomaly. So far we have assumed that the eigenvalues of M_0 are discrete. However, the actual eigenvalue distribution of M_0 would be continuous near $\lambda = -1/3$, and a refinement is of course necessary for (3.17).

Note that we do not have non-vanishing result for (3.17) for all $\widehat{G}$ other than (3.12). Let us consider the following example. By using the formulas (2.12)—(2.14), the LHS of (2.16) is expressed both as $9\boldsymbol{v}_0(1+3M_0)^{-1}\boldsymbol{v}_0$ and as $\boldsymbol{v}_1(1-M_0)^{-1}\boldsymbol{v}_1$, which are respectively the first term and the negative of the second term on the RHS of (3.11) with $\widehat{G}$ replaced with 4. Since both of these two expressions should give the finite value $\ln(3^3/2^4)$ as seen from (2.16), the RHS of (3.11) with $\widehat{G} = 4$ should vanish without any ambiguity. In fact, we have calculated numerically these two quantities by the level truncation to confirm that they both give values close to $\ln(3^3/2^4)$.

This observation for $\widehat{G} = 4$ implies that we need a singularity in $\widehat{G}$ at $M_0 = -1/3$ which would make divergent each of the two terms on the RHS of (3.11) or (3.17) and hence amplify the effect of the breakdown of the degeneracy. In fact, for the genuine $\widehat{G}$ given by (3.12), we have

$$\widehat{G} = -\frac{2\sqrt{3}}{\sqrt{1+3M_0}} - \frac{3\sqrt{3}}{4}\sqrt{1+3M_0} + \cdots, \tag{3.18}$$

around $M_0 = -1/3$.

$1/\sqrt{1+3M_0}$	M_1	$\boldsymbol{v}_0$	$\boldsymbol{v}_1$	$\boldsymbol{t}$
1	-1	0	1	1

Table 2: Degree of singularity for various quantities.

Now, we shall explain how to deform the expressions of G and other quantities interpretable as twist anomaly by respecting their singularity at $M_0 = -1/3$. Of course, calculating these quantities using their original expressions like (3.9) is quite all right. However, by deformations explained below, we can obtain simpler expressions giving the same numerical result. At the same time, the following arguments would be instructive for understanding the phenomenon of twist anomaly. Consider any quantity F given as

$$F = \sum_{\alpha,\beta=0,1} f_{\alpha\beta}\, \boldsymbol{v}_\alpha \mathcal{O}_{\alpha\beta}(M_0, M_1) \boldsymbol{v}_\beta, \tag{3.19}$$

where $f_{\alpha\beta}$ is a scalar coefficient and $\mathcal{O}_{\alpha\beta}(M_0, M_1)$ is a matrix valued function of M_0 and M_1. Let this F vanish like G if we use naively the non-linear relations (2.12)—(2.14). We are allowed, if we wish, to use (3.13) to express $\boldsymbol{v}_0$ in terms of $\boldsymbol{v}_1$ in (3.19). However, we must keep the original ordering among M_0 and M_1. We can make the following simplifications in calculating F. For this purpose, we assign the "degree of singularity" at $M_0 = -1/3$ for various quantities as given table 2. Note that this assignment is compatible with all the non-linear relations (2.12)—(2.16) and the definition (3.10) of $\boldsymbol{t}$. Then, we Laurent-expand $\mathcal{O}_{\alpha\beta}(M_0, M_1)$ with respect to M_0 around $M_0 = -1/3$ by keeping the ordering among the matrices, and count the degree of singularity of each term contributing to (3.19) by summing the degree of the constituents. If the degree of singularity of a term is less than three, we are allowed to freely use all the non-linear relations (2.12)—(2.14) to simplify this term. However, if the degree of singularity is equal to three, we must treat this term as it stands.

For example, following the above rule, G of (3.11) is expressed as

$$G = -\frac{9\sqrt{3}}{32}\, \boldsymbol{v}_1 \left(M_1 \frac{1}{(1+3M_0)^{3/2}} M_1 - \frac{4}{3} \frac{1}{\sqrt{1+3M_0}} \right) \boldsymbol{v}_1 + G_{\mathrm{reg}}, \tag{3.20}$$

where G_{reg} represents the term with degree less than three. Since the whole of (3.20) vanishes by naively using the non-linear relations, we can identify G_{reg} without explicit calculation starting from the original G. Namely, G_{reg} is equal to the negative of the first term of (3.20) calculated by naively using the non-linear relations,

$$G_{\mathrm{reg}} = -\frac{3\sqrt{3}}{32}\, \boldsymbol{v}_1 \sqrt{1+3M_0}\, \boldsymbol{v}_1, \tag{3.21}$$

whose degree of singularity is equal to one.[§] By numerical calculation we find that (3.20) reproduces a value close to $\ln 2$.

We have emphasized above that it is in general dangerous to freely use the non-linear relations (2.12)—(2.14). However, we used them in solving (2.20) for T and (3.7) for $\boldsymbol{t}$. We

[§]Eq. (3.20) with this form of G_{reg} is simply obtained by replacing $\widehat{G}$ in (3.11) with the first singular term of (3.18).

also used non-linear relations in obtaining (3.11) from (3.9). In the rest of this section, we shall discuss the validity of these manipulations.

First, let us reexamine the deformation from (3.9) to (3.11). A possible problem in this deformation is the use of (2.24) which has been obtained by use of non-linear relations. Here we shall consider $(1 - T\mathcal{M})^{-1}$ without using the non-linear relations. To perform it, let us split $1 - T\mathcal{M}$ into the twist-even part S and the twist-odd part A as $1 - T\mathcal{M} = S + A$ with

$$S = \begin{pmatrix} 1 - TM_0 & -T(1 - M_0)/2 \\ -T(1 - M_0)/2 & 1 - TM_0 \end{pmatrix}, \qquad A = \begin{pmatrix} & -TM_1/2 \\ TM_1/2 & \end{pmatrix}. \tag{3.22}$$

Then we have

$$\frac{1}{S + A} = \frac{1}{S} - \frac{1}{S}A\frac{1}{S} + \frac{1}{S}A\frac{1}{S}A\frac{1}{S} - \cdots . \tag{3.23}$$

The inverse of S is simply given as follows because it is given purely in terms of M_0:

$$\frac{1}{S} = \frac{1}{(1 - TM_0)^2 - T^2(1 - M_0)^2/4} \begin{pmatrix} 1 - TM_0 & T(1 - M_0)/2 \\ T(1 - M_0)/2 & 1 - TM_0 \end{pmatrix}. \tag{3.24}$$

Expanding around $M_0 = -1/3$ we have

$$\frac{1}{S} \sim \frac{1}{2\sqrt{3}\sqrt{1 + 3M_0}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{3.25}$$

Therefore, the degree of singularity of S^{-1} is one. Since the twist-odd part A has degree -1, every term in the infinite series (3.23) apparently has the same degree of singularity and it might seem that we cannot simplify the expression (3.23) further. This is, however, not true. Let us first consider the term $S^{-1}AS^{-1}$. The most singular part of $S^{-1}AS^{-1}$ is given by using (3.25) as

$$\frac{1}{S}A\frac{1}{S} \sim \frac{1}{2\sqrt{3}\sqrt{1 + 3M_0}} \frac{M_1}{2} \frac{1}{2\sqrt{3}\sqrt{1 + 3M_0}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{3.26}$$

which vanishes since the product of the three 2×2 matrices is equal to zero. The less singular part in $S^{-1}AS^{-1}$ does not contribute terms with degree of singularity equal to three in G. This argument applies to all the remaining terms in the expansion (3.23). Therefore, in the deformation of (3.9), we are allowed to freely use the non-linear relations for $(1 - T\mathcal{M})^{-1}$ and hence use the expression (2.24).

Finally, let us comment on possible T and $\boldsymbol{t}$ other than those used in this paper. We shall first comment on $\boldsymbol{t}$. One might think that we should solve the original equation (3.7) for $\boldsymbol{t}$ in the level truncation without using the non-linear relations. However, this is impossible if we impose the twist-even condition (3.6) on $\boldsymbol{t}$. In fact, eq. (3.7) and the one obtained from it by multiplying C and using (3.6) are over-deterministic for $\boldsymbol{t}$. As explained in sec. 4.2 of [14], these equations are consistently solved owing to the non-linear relations. On the other hand, the original equation (2.20) for T in the level truncation can have twist-even solutions without using the non-linear relations. We do not know whether such solutions are superior to the conventional one (2.27) in any respects.

4 Potential height problem revisited

If the classical solution Ψ_{c} of VSFT represents a D25-brane, the energy density $\mathcal{E}_{\mathrm{c}}$ of this solution relative to that of the trivial one $\Psi = 0$ should be equal to the D25-brane tension T_{25}. In [14] they obtained the ratio $\mathcal{E}_{\mathrm{c}}/T_{25}$ in a closed form using the Neumann coefficients. They further used the non-linear relations freely to simplify the expression of $\mathcal{E}_{\mathrm{c}}/T_{25}$ and calculated it numerically using the level truncation. The result was not, however, the expected one. Now we know that naive use of the non-linear relations is dangerous. So we shall reexamine the ratio $\mathcal{E}_{\mathrm{c}}/T_{25}$ by taking into account the lesson we learned in the previous section. We shall find that $\mathcal{E}_{\mathrm{c}}/T_{25}$ is expressed in terms of a quantity (called H in [14]) which, like G, vanishes if we freely use the non-linear relations but give a non-vanishing value due to twist anomaly.

Before reexamining the ratio $\mathcal{E}_{\mathrm{c}}/T_{25}$, we shall first summarize the derivation given in [14]. First, the energy density of the solution Ψ_{c} is given by

$$\mathcal{E}_{\mathrm{c}} = -\frac{\mathcal{S}[\Psi_{\mathrm{c}}]}{V_{26}} = \frac{K}{6}\langle\phi_{\mathrm{c}}|\phi_{\mathrm{c}}\rangle = \frac{K}{6}\left(\frac{[\det(1-T\mathcal{M})]^2}{\det(1-T^2)}\right)^{13}\left(\frac{[\det(1-\widetilde{T}\widetilde{\mathcal{M}})]^2}{\det(1-\widetilde{T}^2)}\right)^{-1}, \qquad (4.1)$$

where we have used the equation of motion (2.17) at the second equality. Next, let us calculate the D25-brane tension, which is given in the present convention of $\alpha' = 1$ by $T_{25} = 1/(2\pi^2 g_{\mathrm{o}}^2)$ with g_{o} being the open string coupling constant defined as the three-tachyon on-shell amplitude. Using the tachyon wave function Φ_{t} (3.5), g_{o} is given by

$$\begin{aligned} g_{\mathrm{o}} &= K\,\Phi_{\mathrm{t}}\cdot(\Phi_{\mathrm{t}}*\Phi_{\mathrm{t}})\Big|_{p_1^2=p_2^2=p_3^2=-m_{\mathrm{t}}^2} \\ &= K\mathcal{N}_{\mathrm{t}}^3\,[\det(1-T\mathcal{M}_3)]^{-13}\det(1-\widetilde{T}\widetilde{\mathcal{M}}_3)\,\exp\Bigl\{-\frac{1}{2}\boldsymbol{V}(1-T\mathcal{M}_3)^{-1}TC\boldsymbol{V} \\ &\qquad + \boldsymbol{V}(1-T\mathcal{M}_3)^{-1}\boldsymbol{t}A_0 - \frac{1}{2}A_0\boldsymbol{t}\mathcal{M}_3(1-T\mathcal{M}_3)^{-1}\boldsymbol{t}A_0 - \frac{1}{2}V_{00}(A_0)^2\Bigr\}. \end{aligned} \qquad (4.2)$$

Precisely speaking, we must remove $(2\pi)^{26}\delta^{26}(\sum_r p_r)$ from the second term $K\Phi_{\mathrm{t}}\cdot(\Phi_{\mathrm{t}}*\Phi_{\mathrm{t}})$. The normalization factor $\mathcal{N}_{\mathrm{t}}$ for Φ_{t} in (3.5) is determined by the following requirement that

$\Phi_{\rm t}$ has a canonical kinetic term:

$$\frac{K}{2}\Phi_{\rm t}\cdot\mathcal{Q}_{\rm B}\Phi_{\rm t}\underset{p^2\sim -m_{\rm t}^2}{\sim}-\frac{1}{2}(p^2+m_{\rm t}^2), \tag{4.3}$$

where we have omitted the momentum conservation delta function. We have

$$\mathcal{N}_{\rm t}=\frac{1}{\sqrt{KG}}\left[\det(1-T^2)\right]^{13/2}\left[\det(1-\widetilde{T}^2)\right]^{-1/2}\exp\Bigl(\boldsymbol{t}\,(1+T)^{-1}\boldsymbol{t}\,m_{\rm t}^2\Bigr). \tag{4.4}$$

Collecting all these facts, we find the following expression for the ratio $\mathcal{E}_{\rm c}/T_{25}$:

$$\frac{\mathcal{E}_{\rm c}}{T_{25}}=\frac{\pi^2}{3G^3}\exp(6m_{\rm t}^2H), \tag{4.5}$$

with H defined by

$$\begin{aligned}H=-\frac{2}{(A_0)^2}&\left[-\frac{1}{2}\boldsymbol{V}(1-T\mathcal{M}_3)^{-1}TC\boldsymbol{V}+\boldsymbol{V}(1-T\mathcal{M}_3)^{-1}\boldsymbol{t}A_0-\frac{1}{2}A_0\boldsymbol{t}\mathcal{M}_3(1-T\mathcal{M}_3)^{-1}\boldsymbol{t}A_0\right]\\ &+\boldsymbol{t}(1+T)^{-1}\boldsymbol{t}+V_{00}.\end{aligned} \tag{4.6}$$

All the determinant factors in (4.1), (4.2) and (4.4) have been cancelled out in (4.5) by use of the non-linear relations.

If the classical solution $\Psi_{\rm c}$ represents a single D25-brane, the value of H must be equal to $H=(1/6)\ln(\pi^2/(3(\ln 2)^3))\simeq 0.3817$ (we have used that $m_{\rm t}^2=-1$ and hence $G=\ln 2$). Similarly to G, we can show that H vanishes by freely using the non-linear relations, implying that H is regarded as a twist anomaly which needs careful treatments. In [14] they deformed H (4.6) by using the non-linear relations to obtain another expression ((5.13) of [14]), which gave strange numerical values.¶ However, their manipulations contain forbidden ones in the sense of sec. 3. We have to reexamine H by following the prescription of sec. 3.

To simplify the expression (4.6) for H without changing the ordering among the matrices, note first that $\mathcal{M}_3$ is partially diagonalized as follows:

$$\mathcal{M}_3=W^\dagger\begin{pmatrix}1&&\\&U_+&\\&&U_-\end{pmatrix}W, \tag{4.7}$$

where $U_\pm$ and the unitary matrix W are defined by

$$U_\pm=M_0+\omega^{\pm 1}M_++\omega^{\mp 1}M_-, \tag{4.8}$$

¶Using the general argument of sec. 3, we see that the value of H in the form of (5.13) of [14] is equal to $-G/2$.

$$W=\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{pmatrix}, \tag{4.9}$$

with $\omega=e^{2\pi i/3}$. Using this basis and the fact that $\boldsymbol{V}$ in (2.3) is given in terms of $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ as

$$\boldsymbol{V}=\frac{\boldsymbol{v}_1}{2}\begin{pmatrix}a_0^{(2)}-a_0^{(3)}\\ a_0^{(3)}-a_0^{(1)}\\ a_0^{(1)}-a_0^{(2)}\end{pmatrix}+\frac{3\boldsymbol{v}_0}{2}A_0, \tag{4.10}$$

we obtain an expression of H in terms of smaller matrices:

$$\begin{aligned}H=&-\frac{3}{8}\boldsymbol{v}_1\mathcal{U}^+T\boldsymbol{v}_1+\frac{3\sqrt{3}i}{4}\boldsymbol{v}_1\mathcal{U}^-T\boldsymbol{v}_0+\frac{9}{8}\boldsymbol{v}_0\mathcal{U}^+T\boldsymbol{v}_0-\frac{\sqrt{3}i}{2}\boldsymbol{v}_1\mathcal{U}^-\boldsymbol{t}-\frac{3}{2}\boldsymbol{v}_0\mathcal{U}^+\boldsymbol{t}\\ &+\frac{1}{2}\boldsymbol{t}\Big(U_+(1-TU_+)^{-1}+U_-(1-TU_-)^{-1}\Big)\boldsymbol{t}+\boldsymbol{t}(1+T)^{-1}\boldsymbol{t}+V_{00},\end{aligned} \tag{4.11}$$

with

$$\mathcal{U}^\pm=(1-TU_+)^{-1}\pm(1-TU_-)^{-1}. \tag{4.12}$$

In deriving (4.11), we have used following formula

$$(a_0^{(1)}+\omega a_0^{(2)}+\omega^2 a_0^{(3)})(a_0^{(1)}+\omega^2 a_0^{(2)}+\omega a_0^{(3)})=9, \tag{4.13}$$

which comes from the on-shell condition $(a^{(r)})^2=-2m_{\mathrm{t}}^2=2$. Let us emphasize here again that the matrix ordering has been kept in deriving (4.11) from the original expression (4.6).

We have calculated numerically the value of H using the expression (4.11). The result is given in table 3 (a). Amazingly, the ratio $\mathcal{E}_{\mathrm{c}}/T_{25}$ seems to converge to 2 in contrast to our original expectation of 1. If there are no other subtle points in our analysis, this result implies that the classical solution Ψ_{c} represents the configuration of two D25-branes.

Following the prescription explained in sec. 3, H given by (4.11) can safely be deformed as follows. Taylor-expanding $1-TU_\pm$ around $M_0=-1/3$, we have

$$1-TU_\pm\sim\sqrt{3}\sqrt{1+3M_0}\pm\frac{\sqrt{3}i}{2}M_1. \tag{4.14}$$

Using this, we obtain a simpler expression of H:

$$\begin{aligned}H=&\frac{\sqrt{3}}{4}\boldsymbol{v}_1\left(\frac{1}{\sqrt{1+3M_0}}-3\frac{1}{\sqrt{1+3M_0}}M_1\frac{1}{\sqrt{1+3M_0}}-\frac{9}{2}M_1\frac{1}{1+3M_0}\right)\\ &\times\left(1+\frac{1}{4}M_1\frac{1}{\sqrt{1+3M_0}}M_1\frac{1}{\sqrt{1+3M_0}}\right)^{-1}\boldsymbol{v}_1\end{aligned}$$

L	H	$\mathcal{E}_c/T_{25}$
50	0.27398	1.9089
100	0.27260	1.9247
150	0.27179	1.9341
200	0.27124	1.9405
250	0.27083	1.9453
300	0.27051	1.9490
400	0.27003	1.9546
500	0.26967	1.9589
600	0.26939	1.9622
800	0.26898	1.9670
1000	0.26867	1.9707

(a)

L	H	$\mathcal{E}_c/T_{25}$
50	0.25667	2.1178
100	0.26119	2.0611
150	0.26269	2.0427
200	0.26341	2.0338
250	0.26383	2.0287
300	0.26409	2.0256
400	0.26440	2.0218
500	0.26456	2.0199
600	0.26465	2.0188
800	0.26473	2.0178
1000	0.26476	2.0174

(b)

Table 3: The values of H in the level truncation calculation. In table (a), we used the expression (4.11), while in (b) we used (4.15). For $\mathcal{E}_c/T_{25}$ we used (4.5) with $G = \ln 2$ and $m_t^2 = -1$.

$$+\frac{9\sqrt{3}}{16}\boldsymbol{v}_1 M_1 \frac{1}{(1+3M_0)^{3/2}} M_1 \boldsymbol{v}_1 + H_{\text{reg}}, \tag{4.15}$$

where H_{reg} is the part with degree of singularity less than three. As we did for G_{reg} in (3.20), H_{reg} is given as the negative of the terms on the RHS of (4.15) other than H_{reg} calculated by using naively the non-linear relations and expressed without M_1. Explicitly, we have

$$H_{\text{reg}} = -\frac{\sqrt{3}}{16}\boldsymbol{v}_1 \frac{(1+3M_0)^{3/2}}{5-M_0}\boldsymbol{v}_1. \tag{4.16}$$

This simpler form (4.15) of H should give the same value as the original one. In fact, the result of level truncation calculation presented in table 3 (b) confirms this expectation.

5 Summary and future problems

In this paper we have presented an interpretation as twist anomaly to G and H expressing the physical observables in VSFT expanded around a classical solution. We have reexamined the potential height problem for the solution and obtained a result indicating that it represents the configuration of two D25-branes. Let us finish this paper by presenting future problems.

- We have obtained in this paper a numerical result that $\mathcal{E}_c/T_{25} = 2$. Though this result is not obviously strange, it is not a natural one. In arriving at the formula (4.5) for the ratio $\mathcal{E}_c/T_{25}$, all the determinant factors in (4.1) and (4.4) have been cancelled out among them. However, since the eigenvalue distribution of M_0 and T extends to $-1/3$ and -1, respectively, the cancellation of the determinants contains indefinite quantities like 0/0. In fact, numerical analysis of the determinants given in [14] indicates that the cancellation among the determinants is subtle. We have to clarify these points for obtaining the final answer to the ratio $\mathcal{E}_c/T_{25}$.

- In this paper, we have calculated G and H numerically by using the level truncation. It is of course desirable to develop a method to calculate them analytically. For example, exact expression of eigenvalue distribution function for M_0 would be a first step toward this subject.

- The matrix T used in this paper has been obtained by freely using the non-linear relations among M_α on the original equations for T. As mentioned at the end of sec. 3, there are other candidate solutions for T obtained without using the non-linear relations. We have to examine whether such solutions give different results for physical quantities.

- The most important and interesting problem for VSFT is to show that the perturbation theory expanded around the trivial configuration $\Psi = 0$ reproduces pure closed string theory. As seen in this paper, physical observables in VSFT are interpretable as twist anomaly. Closed string might also emerge as a twist anomaly.

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